Equivalence of driven and aging fluctuation-dissipation relations in the trap model

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We study the nonequilibrium version of the fluctuation-dissipation (FD) relation in the glass phase of a trap model that is driven into a nonequilibrium steady state by external "shear." This extends our recent study of aging FD relations in the same model, where we found limiting, observable independent FD relations for "neutral" observables that are uncorrelated with the system's average energy. In this work, for such neutral observables, we find the FD relation for a stationary weakly driven system to be the same, to within small corrections, as for an infinitely aged system. We analyze the robustness of this correspondence with respect to non-neutrality of the observable, and with respect to changes in the driving mechanism.

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I. INTRODUCTION

Glasses relax very slowly at low temperatures. They therefore remain out of equilibrium for long times and exhibit aging [1]: the time scale for response to an external perturbation (or for the decay of correlations) increases with the "waiting time" $t_{\rm w}$ since the system was quenched to the low temperature, and thus eventually far exceeds the experimental time scale. Time translational invariance (TTI) is lost. As a result of this dynamical sluggishness, glassy systems are highly susceptible to external driving, even when the driving rate $\dot{\gamma}$ is small. One example of $\dot{\gamma}$ is shear rate in a rheological system. Typically, steady driving interrupts aging and restores a nonequilibrium steady (TTI) state in which the time scale defined by the inverse driving rate plays a role analogous to the waiting time $t_{\rm w}$ of the aging regime [2–8].

Aging and driven glasses, in general, violate the equilibrium fluctuation-dissipation theorem (FDT) [9]. Consider the autocorrelation function for a generic observable m, defined as $C(t,t_{\rm w})=\langle m(t)m(t_{\rm w})\rangle-\langle m(t)\rangle\langle m(t_{\rm w})\rangle$. The associated step response function $\chi(t,t_{\rm w})=\int_{t_{\rm w}}^{t}dt'R(t,t')$ tells us how m responds to a small step $h(t)=h\Theta(t-t_{\rm w})$ in its conjugate field h. In *equilibrium*, $C(t,t_{\rm w})=C(t-t_{\rm w})$ by TTI (similarly for χ), and the FDT reads $-(\partial/\partial t_{\rm w})\chi(t-t_{\rm w})=R(t-t_{\rm w})=(1/T)(\partial/\partial t_{\rm w})C(t-t_{\rm w})$, where $R(t-t_{\rm w},t_{\rm w})=(\delta\langle m(t)\rangle/\delta h(t_{\rm w}))|_{h=0}$ is the impulse response function and T is the thermodynamic temperature. (We set $k_{\rm B}=1$.) A parametric 'FD plot' of χ vs C is thus a straight line of slope $-1/T$.

Out of equilibrium, violation of FDT is measured by an FD ratio, $X(t,t_{\rm w})$, defined through [10,11]

$$-\partial/\partial t_{\rm w}\chi(t,t_{\rm w})=R(t,t_{\rm w})=\frac{X(t,t_{\rm w})}{T}\partial/\partial t_{\rm w}C(t,t_{\rm w}). \quad (1)$$

In aging systems, violation ($X\neq 1$) can persist even at long times, indicating strongly nonequilibrium behavior even though one-time observables such as entropy may have settled to essentially stationary values. Similarly, driven glassy systems can violate FDT even in the limit of weak driving.

Remarkably, the FD ratio in several aging mean field models [10,11] assumes a special form at long times. Taking $t_{\rm w}\to\infty$ and $t\to\infty$ at constant $C=C(t,t_{\rm w})$, $X(t,t_{\rm w})\to X(C)$ becomes a (nontrivial) function of the single argument C. If the equal-time correlator $C(t,t)$ also approaches a constant C_0 for $t\to\infty$, it follows that:

$$\chi(t,t_{\rm w})=\frac{1}{T}\int_{C(t,t_{\rm w})}^{C_0}dC\,X(C). \quad (2)$$

Graphically, this limiting nonequilibrium FD relation is obtained by plotting χ vs C for increasingly large times. From the slope $-X(C)/T$ of the limit plot an effective temperature [12] $T_{\rm eff}(C)=T/X(C)$ can be defined. Throughout this paper, we absorb the factor T into the response function so that an equilibrium FD plot has slope -1.

In the most general aging scenario, a system displays dynamics on several characteristic time scales, each with its own functional dependence on $t_{\rm w}$. If these time scales become infinitely separated as $t_{\rm w}\to\infty$, they form a set of distinct "time sectors." In mean field, $T_{\rm eff}(C)$ is a *constant* within each sector [11], and *independent* of the observable m used to construct the FD plot. These properties have also been observed in some lower-dimensional (non mean field) systems [1,13].

Cugliandolo *et al.* [12] proposed that an equivalent limiting nonequilibrium FD relation should hold in slowly *driven* glassy systems ($\dot{\gamma}\to 0$), and that the corresponding effective temperatures $T_{\rm eff}(C,\dot{\gamma}\to 0)$ and $T_{\rm eff}(C,t_{\rm w}\to\infty)$ should coincide. Although this is believed to apply widely among driven glasses, the evidence supporting it, to date, is limited to the two detailed studies of Berthier *et al.* in mean field [2] and in simulations of sheared Lennard Jones particles, initially in Ref. [14] and later, with a study of observable independence, in Refs. [16,15]. We note that FDT has also been studied in a

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driven phase separating model ($T<T_c$, where T_c is the critical temperature) with N nonconserved order parameters in the large N limit [17], although here the drive does not interrupt aging. In this study, in the time sector with stationary dynamics, the FD plot is of trivial equilibrium form for $\dot{\gamma}=0$ but a nontrivial curve for $\dot{\gamma}>0$; in the aging sector, the response function is a constant, giving a flat FD plot.

In this paper, therefore, we study FDT in the driven regime of Bouchaud's trap model [18–20,7], for which correlation and response functions can be calculated exactly. This will allow a comparison with our recent study [21] of the aging trap model, where we found limiting FD relations that are observable independent for "neutral observables" that are uncorrelated with the system's energy. This is consistent with the mean field work, for which observables are usually defined in terms of random couplings, uncorrelated with the average energy. (In coarsening models, similar arguments have been used to exclude observables correlated with the order parameter [22].) Surprisingly, however, we found the FD plot to be a continuous curve even though the model has just one time sector, with relaxation times $O(t_{\rm w})$. Although this finding is apparently at odds with the mean field predictions, it is likely to result from the fact that the trap model has a broad distribution of relaxation times (all within its single time sector). We will return to this point in the conclusion.

In what follows, our central result will be that *the same* nontrivial FD relation is found (to within logarithmic corrections), even when the trap model is weakly driven according to the mechanism proposed in Ref. [7]. Although the curvature of the FD plot obviously excludes a constant effective temperature, our finding is still consistent with the predictions of Cugliandolo *et al.* insofar as *the relationship between correlation and response is the same for aged and weakly driven glasses*. This finding is nontrivial since the shapes of the relaxation spectra for the aging and weakly driven trap models differ strongly from each other.

We start (Sec. II) by defining the trap model and summarising the aging FD predictions of Ref. [21]. We then (Sec. III) derive exact expressions for the autocorrelation and response functions for an arbitrary observable m in the steadily driven regime. Using these, we calculate the limiting driven FD relation, for neutral observables that are uncorrelated with the average energy. We show that this relation is the same (to within logarithmic corrections) as its aging counterpart (Sec. IV). We then (Sec. V) consider robustness of this correspondence with respect to (i) changes in the driving mechanism and (ii) non-neutrality of the observable, before concluding.

II. THE TRAP MODEL; DRIVING

The trap model [18] comprises an ensemble of uncoupled particles exploring a spatially unstructured landscape of (free) energy traps by thermal activation. The tops of the traps are at a common energy level and their depths E have a "prior" distribution $\rho(E)$ ($E>0$). A particle in a trap of depth E escapes on a time scale $\tau(E)=\tau_0\exp(E/T)$ and hops into another trap, the depth of which is drawn at random from $\rho(E)$. The probability, $P(E,t)$, of finding a randomly chosen particle in a trap of depth E at time t thus obeys

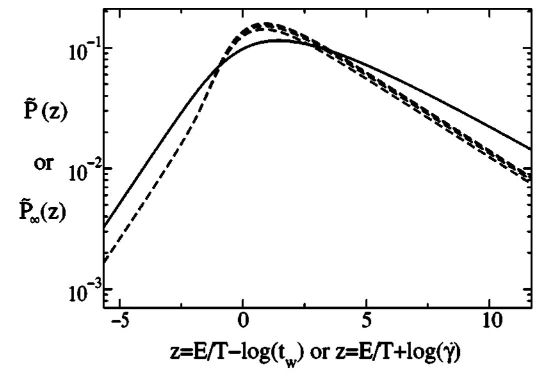

FIG. 1. Scaled energy distributions for (i) the aging model at $t_{\rm w}=10^3$, 10^4, 10^5, 10^6 (dashed lines) and (ii) the driven model for $\dot{\gamma}=10^{-3}$, 10^{-4}, 10^{-5}, 10^{-6} (solid lines; different $\dot{\gamma}$ values indistinguishable). The temperature $T=0.3$.

$$\partial_t P(E,t) = -\tau^{-1}(E)P(E,t) + Y(t)\rho(E) \qquad (3)$$

in which the first (second) term on the RHS (right-hand side) represents hops out of (into) traps of depth E, and $Y(t)=\langle \tau^{-1}(E)\rangle_{P(E,t)}$ is the average hopping rate.

For the specific choice of prior distribution $\rho(E)\sim\exp(-E/T_{\rm g})$, the model shows a glass transition at a temperature $T_{\rm g}$. This is seen as follows. At a temperature T, the equilibrium state (if it exists) is $P_{\rm eq}(E)\propto\tau(E)\rho(E)\propto\exp(E/T)\exp(-E/T_{\rm g})$. For temperatures $T\leqslant T_{\rm g}$ this is unnormalizable, and cannot exist; the lifetime averaged over the prior $\langle\tau\rangle_\rho$ is infinite. Following a quench to $T\leqslant T_{\rm g}$, the system never reaches a steady state, but instead ages. This can be seen from the time evolution of $P(E,t_{\rm w})$, which can be obtained exactly from Eq. (3) (with $Y(t_{\rm w})$ determined self-consistently by enforcing normalization of $P(E,t_{\rm w})$ [18,19]). At large times $t_{\rm w}\to\infty$ a scaling limit is reached in which $P(\tau,t_{\rm w})=[T/\tau(E)]P(E,t_{\rm w})$ is concentrated entirely on traps of lifetime $\tau=O(t_{\rm w})$: the scaling distribution $\tilde{P}(z)=TP(E)$, where $z=E/T-\ln(t_{\rm w})$ is shown in Fig. 1. The model thus has just one characteristic time scale, which grows linearly with the age $t_{\rm w}$. [In contrast, for $T>T_{\rm g}$ all relaxation processes occur on time scales $O(\tau_0)$.] In what follows, we rescale all energies such that $T_{\rm g}=1$, and times so that $\tau_0=1$.

Driving was first incorporated into the model in order to study the rheology of "soft glassy materials." Although we are not directly interested in rheology here, we use the same driving rules which are defined as follows. Each particle is assigned its own local elastic "strain" l, with a corresponding "stress" kl. Each time the particle hops, l is set to zero. Between hops, $\dot{l}=\dot{\gamma}$ where $\dot{\gamma}$ is the rate of external driving ("straining"). A particle in a trap of depth E strained by l sees a reduced effective energy barrier $E-\frac{1}{2}kl^2$, so that

$$D_t P(E,l,t) = -\tau^{-1}(E)e^{kl^2/2T}P + Y(t)\rho(E)\,\delta(l). \qquad (4)$$

In this equation, $\tau^{-1}(E)e^{kl^2/2T}$ is the strain-enhanced counterpart of the "bare" activation rate $\tau^{-1}(E)$ defined above for the undriven model, and D_t is the convected derivative $D_t=\partial_t+\dot{\gamma}\partial_l$. Equation (4) (integrated on l) reduces to Eq. (3) for $\dot{\gamma}=0$, as required. In the following we rescale l such that $k=1$.

For steady driving ($\dot{\gamma}=$const) aging is interrupted, and a TTI steady state is restored. To write down the steady-state distribution, define $S(l_1,l_2,E)$ as the probability for a particle that starts off with strain l_1 and in a trap of depth E not to hop until its strain has reached l_2. From Eq. (4), this is

$$S(l_1,l_2,E)=\exp\left[-\frac{1}{\tau(E)\dot{\gamma}}\int_{l_1}^{l_2}dl\,\exp\left(\frac{l^2}{2T}\right)\right]. \qquad (5)$$

In terms of this quantity, the steady-state distribution of Eq. (4) can then be written as

$$P_\infty(E,l)=\frac{Y_\infty}{\dot{\gamma}}\rho(E)S(0,l,E). \qquad (6)$$

Here $Y_\infty\sim\dot{\gamma}^{1-T}$ is the steady-state average hopping rate, which can be determined self-consistently by enforcing normalization of $P_\infty(E,l)$. [This is most conveniently done by changing variables to $\tau=\exp(E/T)$ which gives $\rho(\tau)\sim\tau^{-1-T}$.] In the limit $\dot{\gamma}\to 0$, the energy distribution $P_\infty(E)=\int dl\,P_\infty(E,l)$ approaches a scaling limit in which all relaxation times are $O(1/\dot{\gamma})$. The scaling distribution $\tilde{P}_\infty(z)=TP_\infty(E)$, where $z=E/T+\ln(\dot{\gamma})$ differs strongly from its aging counterpart $\tilde{P}(z)$, as shown in Fig. 1.

III. CORRELATION AND RESPONSE

FDT can be studied by assigning to each trap, in addition to its depth E, a value for an (arbitrary) observable m [21]. The trap population is then characterized by the joint prior distribution $\sigma(m|E)\rho(E)$, where $\sigma(m|E)$ is the distribution of m across traps of given fixed energy E. The dynamics then obey

$$D_tP(E,m,l,t)=-\tau^{-1}(E,m)e^{kl^2/2T}P + Y(t)\rho(E)\delta(l)\sigma(m|E), \qquad (7)$$

where the activation times are modified by a small field h conjugate to m as $\tau(E,m)=\tau(E)\exp(mh/T)$. This particular form of $\tau(E,m)$ is one of several possible choices that all maintain detailed balance under zero-driving conditions [20,23]. We adopt it because, in the spirit of the unperturbed model ($h=0$), it ensures that the jump rate between any two states depends only on the initial state, and not the final one.

In Ref. [21] we derived exact expressions for the two-time autocorrelation and step response functions, $C(t,t_w)$ and $\chi(t,t_w)$ in the aging regime ($t_w\to\infty$, $t\to\infty$ at fixed t/t_w) of the undriven model ($\dot{\gamma}=0$), following a quench into the glass phase at time $t_w=0$ from an initially infinite temperature. Each comprises two components that depend separately

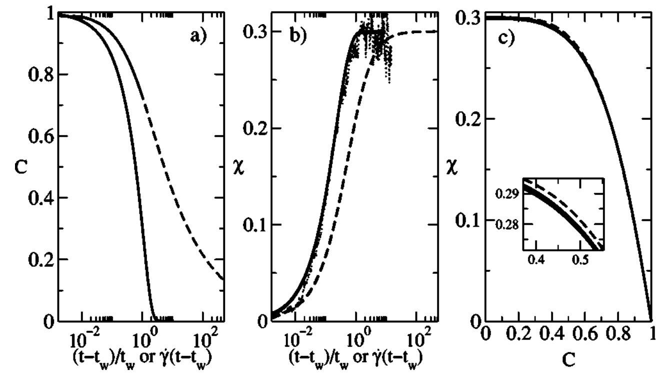

FIG. 2. (a) correlator and (b) response vs. scaled time for the neutral observable $\Delta^2(E)=1$ in the driven model (solid lines) and aging model (dashed lines), calculated from the exact analytical expressions in the text and in Ref. [21]. For the aging case, waiting times $t_w=10^3$, 10^4, 10^5, 10^6 are shown (but are indistinguishable from each other); for the driven case shear rates $\dot{\gamma}=10^{-3}$, 10^{-4}, 10^{-5}, 10^{-6} are shown (also indistinguishable). As an independent check, we also show the driven correlator and response at $\dot{\gamma}=10^{-3}$ calculated from waiting time Monte Carlo simulation. For C this is indistinguishable from the exact results for C in (a); for χ it appears as the jagged line in (b). (c) FD plots of correlator vs. response for the driven case (solid lines) and aging case (dashed lines) constructed from the exact results of (a),(b); the inset is zoomed on a small region of the main plot. For the driven case (solid lines), driving rate decreases downwards at fixed C. The temperature $T=0.3$.

upon the functional forms of the mean $\overline{m}(E)$ and variance $\Delta^2(E)$ of the distribution $\sigma(m|E)$. For the purposes of this paper we are interested only in observables with $\overline{m}(E)=0$ and (within these) mainly the neutral observable for which the variance is uncorrelated with energy $\Delta^2(E)=\text{const}\equiv 1$. In this case, in the simultaneous limit $t_w\to\infty$ with $t\to\infty$, $C(t,t_w)$ and $\chi(t,t_w)$ depend on time only through the scaling variable $(t-t_w)/t_w$ as shown in Figs. 2(a), 2(b). The corresponding FD plot is shown in Fig. 2(c).

In the steadily driven regime, TTI is restored: C and χ do not depend explicitly upon the waiting time t_w but only on the measurement interval $t-t_w$, so we set $t_w=0$ without loss of generality. For observables with $\overline{m}(E)=0$, the autocorrelation function is exactly

$$C(t,\dot{\gamma})=\int_{\dot{\gamma}t}^{\infty}dl\int_0^\infty dE\,\Delta^2(E)P_\infty(E,l). \qquad (8)$$

This can be understood as follows. When any particle hops, its new value of m is uncorrelated with the old one. At time t, therefore, only those particles that have not hopped since $t=0$ can contribute to the correlator, with weight $\Delta^2(E)$. The fraction of such particles which had strain l and trap depth E at time $t=0$ is $P_\infty(E,l)S(l,l+\dot{\gamma}t,E)$. From Eq. (4) this equals $P_\infty(E,l+\dot{\gamma}t)$ and integration on l and E gives the result, Eq. (8). [Alternatively, Eq. (8) can be understood by recalling the driving dynamics: upon any hop, each particle resets its local strain to zero; between hops, the local strain affinely follows the applied one. Therefore, the particles that have not hopped since $t=0$ are just those that have strains $l \geqslant \dot{\gamma} t$.]

The corresponding switch-on response function has contributions from both hopped and unhopped particles. The contribution from hopped particles can be expressed as an integral over the last hop time of each particle, t':

$$\chi_{\rm hop}(t,\dot{\gamma}) = \partial_h|_{h=0} \int_{-\infty}^{\infty} dm \int_0^{\infty} dE \int_0^t dt' \, Y(h,t') m \sigma(m|E) \rho(E) \exp\left[-\frac{1}{\dot{\gamma}\tau(E,m)} \int_0^{\dot{\gamma}(t-t')} ds \, \exp\left(\frac{s^2}{2T} \right) \right]. \tag{9}$$

In this expression, $Y(h,t) = Y_\infty + O(h)$ is the fraction of particles that last hopped at time t'. Of these, a proportion $\sigma(m|E)\rho(E)$ chose energy E and magnetization m. The subsequent survival probability over the interval $t' = 0 \cdots t$ is encoded by the exponential factor. In principle, the differentiation on h has two contributions: one from the factor $\tau(E,m)$ in the exponential, and the other from $Y(h,t)$. However, the second gives zero, since $\int_{-\infty}^{\infty} m \sigma(m|E) = 0$ for the zero-mean variables considered here. Adding the contribution from particles that have not hopped since $t=0$, and doing some manipulation, we find finally the exact result

$$\chi(t,\dot{\gamma}) = \int_0^\infty dE \int_0^\infty dl \, \frac{\Delta^2(E)}{\dot{\gamma}\tau(E)} [P_\infty(E,l) - P_\infty(E,l+\dot{\gamma}t)] \int_0^l ds \, \exp\left(\frac{s^2}{2T} \right) \tag{10}$$

(into which we have absorbed a factor T, as described above).

IV. FD PLOTS

Using the exact expressions of Eqs. (8) and (10), we calculated $C(t,\dot{\gamma})$ and $\chi(t,\dot{\gamma})$ numerically for the neutral observable, $\Delta^2(E)=1$. As an independent check, we calculated each quantity by direct simulation, using a waiting time Monte Carlo technique. The results are shown in Figs. 2(a), 2(b). In the limit $\dot{\gamma} \to 0$, $t \to 0$ at fixed $\dot{\gamma}t$, $C(t,\dot{\gamma})$ and $\chi(t,\dot{\gamma})$ depend on $\dot{\gamma}$ and t only through the scaling variable $\dot{\gamma}t$. Analytically, this can be seen for the correlator by substituting Eq. 6 into Eq. 8 and integrating on dE by changing variables to τ, as described above. The $\dot{\gamma}$ dependence from the integral exactly cancels the prefactor $Y_\infty/\dot{\gamma}$ so that the only dependence on $\dot{\gamma}$ and t appears through the scaling variable $\dot{\gamma}t$ in the limit of the integral on l. A similar argument applies to the response function.

The scaling functions $C(\dot{\gamma}t)$ and $\chi(\dot{\gamma}t)$ both differ strongly from their aging counterparts [compare the solid and dashed lines in Figs. 2(a), 2(b)]. This is to be expected, due to the obvious difference between the (scaled) energy distributions of the driven and undriven models. Remarkably, however, the driven FD relation $\chi(C)$ is strikingly similar to its undriven counterpart [Fig. 2(c)]. Both start with a slope $-X(C=1) \equiv \chi'(C=1) = -1$ (thus reproducing the equilibrium FD form in this limit) and finish with a slope $\chi'(C=0)=0$ at an intercept $\chi(C=0)=T$. (We have confirmed these features analytically as well as numerically.) Even between these limits, these is little discernible difference between the aging and driven FD plots. This nontrivial result is consistent with the predictions of Cugliandolo *et al.*, that the relationship between correlation and response should be the same in weakly driven and old aging glassy systems.

In the inset of Fig. 2(c), we show an expanded region of the main FD plot. Despite the striking similarity of the aging and driven FD relations, our numerics do nonetheless suggest a small discrepancy. To investigate this further, we examined the behavior of $X \equiv -\chi'(C)$ in the limit $C \to 0$. Setting $(t-t_{\rm w})/t_{\rm w} = u$ for the aging model, we found $C \sim u^{-T}$ and $X \sim u^{-1}$, hence $X \sim C^{1/T}$ as $u \to \infty$. Setting $\dot{\gamma}t = v$ for the driven model, we found $C \sim v^{T-1} \exp(-v^2/2)$ and $X \sim \exp[-v^2/2T]$, hence $C \sim X^T [\ln(1/X)]^{(T-1)/2}$ as $v \to \infty$. Therefore, the driven and aging FD plots are indeed equivalent in the limit $C \to 0$, but only to within minor logarithmic corrections. This explains the slight discrepancy seen in our numerical data.

V. ROBUSTNESS UNDER CHANGE OF DRIVING MECHANISM; NON-NEUTRAL OBSERVABLES

We now investigate the robustness of the equivalence between driven and aging FD relations with respect to (i) nonneutrality of the observable m, and (ii) changes in driving mechanism. We start with (i), considering observables for which $\Delta^2(E) = \exp(nE/T)$ (which defines n), though still with $\overline{m}(E) = 0$. In this case, the overall amplitude (initial value) of the correlator depends explicitly on the waiting time (or driving rate), even in the aging (or weakly driven) limit. In order to obtain a limiting FD plot, the correlator and response must be normalized by the initial value of the correlator. In the driven regime, therefore, we now plot $\tilde{\chi}(t,\dot{\gamma}) \equiv \chi(t,\dot{\gamma})/C(0,\dot{\gamma})$ versus $\tilde{C}(t,\dot{\gamma}) = C(t,\dot{\gamma})/C(0,\dot{\gamma})$, with t as the plotting parameter. In the aging case, for these nonneutral observables, care must be taken in constructing the FD plot. For neutral observables, the usual prescription is to plot $\chi(t,t_{\rm w})$ versus $C(t,t_{\rm w})$ with t as the plotting parameter. For non-neutral observables, however, we see from Eq. (1) that the slope of the FD plot is only guaranteed to coincide with X if we plot $\tilde{\chi}(t,t_{\rm w}) \equiv \chi(t,t_{\rm w})/\chi(t,t)$ versus $\tilde{C}(t,t_{\rm w})$

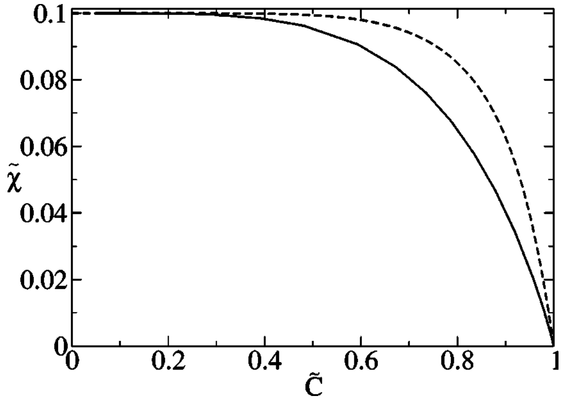

FIG. 3. FD plots for (i) the aging model with $t=10^6$, 10^7 (dashed lines; different t values indistinguishable), and (ii) the driven model with $\dot{\gamma}=10^{-3}$, 10^{-4}, 10^{-5}, 10^{-6} (solid lines; different $\dot{\gamma}$ values indistinguishable) for a non-neutral observable with $\overline{m}(E)=0$, $\Delta^2(E)=\exp(nE/T)$ for $n=0.2$. The temperature $T=0.3$.

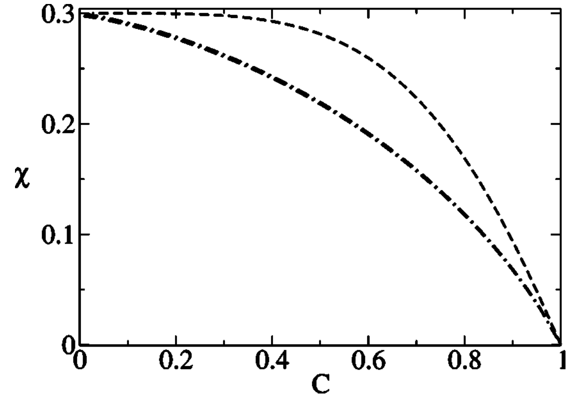

FIG. 4. FD plots for the neutral observable with $\overline{m}(E)=0$, $\Delta^2(E)=1$ for (i) the aging model with $t=10^3$, 10^4, 10^5, 10^6 (dashed lines; different t values indistinguishable) and (ii) a driven model with the alternative driving dynamics of Eq. (11) for $\dot{\gamma}=10^{-3}$, 10^{-4}, 10^{-5}, 10^{-6} (dot-dashed lines). The temperature $T=0.3$.

$\equiv C(t,t_{\rm w})/C(t,t)$ *with* $t_{\rm w}$ *as the plotting parameter*. (This coincides with the usual prescription for neutral observables, as required.) These normalized FD plots are shown for $n=0.2$ in Fig. 3, and are seen to differ strongly from each other: equivalence of aging and driven FD relations does not hold in the trap model for non-neutral observables.

Finally, we consider robustness of the equivalence of aging and driven FDT with respect to a change in the driving mechanism. To do this, we consider a different driving mechanism which adds an additional hopping process to the trap model whose rate $\dot{\gamma}$ is independent of trap depth. This just has the effect of normalizing each hopping rate according to $1/\tau \to 1/\tau + \dot{\gamma}$. The dynamics are now

$$\partial_t P(E,m,t) = -\left[\frac{1}{\tau(E,m)} + \dot{\gamma}\right] P + Y(t)\,\rho(E)\sigma(m|E) \tag{11}$$

with a steady-state energy distribution given by

$$P_\infty(E) = \frac{Y_\infty \rho(E)}{\tau^{-1}(E) + \dot{\gamma}}. \tag{12}$$

The calculation of correlation and response functions is now trivial since all survival probabilities are simple exponentials; one finds

$$C(t,\dot{\gamma}) = \int_0^\infty dE\,\Delta^2(E) P_\infty(E) \exp\left[-\left(\frac{1}{\tau(E)} + \dot{\gamma}\right)t\right], \tag{13}$$

and

$$\chi(t,\dot{\gamma}) = \int_0^\infty dE\,\Delta^2(E) P_\infty(E) \frac{1}{1+\dot{\gamma}\tau(E)} \times \left\{1 - \exp\left[-\left(\frac{1}{\tau(E)} + \dot{\gamma}\right)t\right]\right\}. \tag{14}$$

The FD plot for the neutral observable $\Delta^2(E)=1$ is given in Fig. 4, and seen to differ strongly from the corresponding FD plot in the undriven model. Hence, the equivalence of the aging and driven FD relations is not preserved for this change of driving mechanism. However, this is likely to be a consequence of the fact that this second choice of driving mechanism does not in fact violate detailed balance, but merely renormalises all the jump rates.

VI. SUMMARY AND CONCLUSION

In this paper we studied the nonequilibrium FDT in the glass phase of Bouchaud's trap model [18–20,7], extended to incorporate the nonlinear driving mechanism of Ref. [7]. After deriving exact expressions for the correlation and response functions of a generic observable, m, we compared the FD relation for a system driven steadily at rate $\dot{\gamma}\to 0$ with that for an aging system at waiting time $1/\dot{\gamma}$. For "neutral" observables that are uncorrelated with the system's average energy, the driven and aging FD relations are the same, to within minor logarithmic corrections. This correspondence does not apply to non-neutral observables. Finally, we considered an alternative driving mechanism that renormalises all the hopping rates according to $1/\tau \to 1/\tau + \dot{\gamma}$. In this case, the aging and driven FD relations differ strongly, even for neutral observables. Although this is apparently at odds with our central result, it is likely to result from the fact that this trivial driving mechanism does not violate detailed balance. Further research is certainly needed, however, to understand whether other conditions are required on driving mechanisms in order to get steady-state behavior related to that of undriven aging systems.

We return finally to address the fact that the FD relations are rounded in this model, thus excluding a single effective temperature within the aging or driven time sectors (with relaxation times $O(t_w)$ or $O(\dot{\gamma})$, respectively). Similarly ''rounded'' FDT plots have recently been found in coarsening models at criticality [24]; the limiting value $-X_\infty$ of the slope for $C\rightarrow 0$ was there shown to be a universal amplitude ratio. It is possible that at least this X_∞ could define a sensible $T_{\rm eff}$, and in fact both our limiting FDT plots (for the first driving mechanism) share a common value $X_\infty=0$.

In conclusion, the FD relation between correlation and response is the same, to within logarithmic corrections, in the aging and driven trap models. We suggest that the limiting value $-X_\infty$ of the slope for $C\rightarrow 0$ could be used to define an effective temperature. Further work is needed to delineate more fully the class of finite-dimensional driven glassy models that exhibit this behavior.

Note added. We have recently become aware that Ludovic Berthier has studied FDT in the driven EA model, and found FD plots the same as in the aging model, with agreement between two different observables (unpublished).

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